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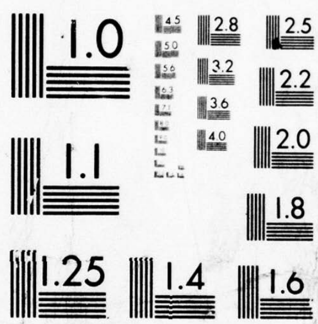

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PROJECT SQUID

TECHNICAL REPORT UC-1-PU

MECHANISMS OF DECAY OF LAMINAR AND TURBULENT VORTICES

by

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Mechanisms of decay of laminar and turbulent vortices

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Dynamics of an infinitely long one-dimensional vortex and a swirl are compared with dynamics of a semi-infinitely long trailing vortex and trailing swirl. With increasing distance, the change in axial velocity difference between the core of the trailing-vortex and the surrounding region causes radial convection and some associated axial convection of angular momentum. In laminar or turbulent trailing vortices, this is the dominant mechanism for decrease in velocities of swirl in the core and corresponding growth of the core. Axial velocity difference between the core of the trailing-vortex and the surrounding region is necessary for the sustenance of turbulence in the vortex core. A theory of turbulent trailing-vortex is developed on the basis of these mechanisms and the results are compared with experimental observations.

1. Introduction

There are a variety of theories and views about unconfined and semi-infinitely or infinitely long turbulent swirls and vortices. These theories are neither confirmed nor refuted by experimental investigations, since very few experiments exist where the effects of initial conditions and extraneous influences have been minimized. We critically examine the assumptions underlying these theories and show that many of these are far-fetched. We find correct physical processes. The results of analysis based on these physical processes are compared with the most recently available experiments.

There are four distinct types of flows which are relevant here. An infinitely long LINE SWIRL with a single velocity component $u_\theta(r,t)$, where r is the radial distance from the swirl axis z, subscript θ refers to the angular coordinate, and t is the time. Its angular momentum ρM per unit axial distance is constant and independent of time, where

$$M = 2\pi \int_0^\infty u_\theta(r,t) r^2 dr , \tag{1}$$

and ρ is the density of the fluid. A prime, as u'_θ, will be used to denote the fluctuating part of a quantity. Otherwise, the symbol represents either a quantity in a laminar flow or its mean value in a turbulent flow.

The second flow is an infinitely long LINE VORTEX with one velocity component $u_\theta(r,t)$ such that $u_\theta = \Gamma_0/2\pi r$ for large r with zero velocity at r = 0. Its angular momentum per unit distance along the swirl axis is infinite and its rate of change of angular momentum is finite depending on viscosity. It is further assumed that $\partial\Gamma^2/\partial r>0$ where $\Gamma = 2\pi r u_\theta$.

The third flow is a TRAILING SWIRL with three velocity components $u_z(r,z)$, $u_\theta(r,z)$, and $u_r(r,z)$, where z is the distance along the axis of the trailing swirl measured from its origin. It may be produced by rotating vanes with the axis of rotation parallel to the prevailing uniform steady flow u_o. The vanes may add to or absorb some axial momentum of the prevailing flow. Thus we have a swirl which may have a coaxial jet or a wake. As $r \to \infty$ $u_\theta = u_r = 0$, and $u_z = u_o$. The flux of angular momentum

$$A = 2\pi\rho_o \int_0^\infty u_z(r,z)\, u_\theta(r,z)\, r^2\, dr \tag{2}$$

is independent of z.

The fourth flow is a TRAILING VORTEX produced at the tip of a semi-infinite lifting wing in the presence of a prevailing mean flow. The trailing vortex has some similarity to a trailing swirl having three velocity components, with the important difference that as $r \to \infty$, $u_r = 0$, $u_\theta = \Gamma_o/2\pi r$, $u_z = u_o$ where Γ_o is the total circulation. The flux of angular momentum is infinite, but its change

$$dA/dz = 2\pi\rho_o \int_0^\infty \frac{\partial}{\partial z}(u_z u_\theta)\, r^2\, dr \tag{3}$$

is finite and depends on z. It is further assumed that $\partial\Gamma^2/\partial r > 0$.

Initially, in all these flows vorticity is assumed to be confined to a region of small radius and moments of all orders of any vorticity component with respect to z-axis exist. The cores of these flows are regions where most of vorticity is located. This paper is concerned with the growth of these cores or the spread of initially concentrated vorticity due to viscous and turbulent processes.

In analyzing the third and fourth flows, it is invariably assumed that u_r may be neglected, $u_z = u_o$, and z is replaced by tu_o, thus reducing the trailing swirl to the line swirl and the trailing vortex to a line vortex. We indicate below and subsequently show in detail that this approximation is invalid.

As a trailing swirl or a trailing vortex develops, swirl velocities u_θ decrease with the downstream distance z. Since the pressure at large r is constant, near the axis of a swirl or a vortex this leads to $\partial p/\partial z > 0$ where p is the pressure. This implies divergence of the cores of these flows and an axial velocity difference between the cores of these flows and the surrounding regions that have three important effects on the dynamics of these trailing flows which are absent in a line swirl and a line vortex. (1) Linear and non-linear stability analyses (Uberoi et al. 1972; Narain and Uberoi 1973) show that a difference of axial velocities between the core and the surroundings destabilizes swirling flows which otherwise could be stable. (2) There may be significant and sometimes dominant radial and associated axial convection of angular momentum. (3) The range of downstream distances over which dynamic self-similarity exists may be limited. The importance of these effects decreases with decreasing rate of spread of these flows. However, in the study of trailing flows with swirl the emphasis is on their rates of growth rather than on the final stages where they have practically ceased to grow.

2. Laminar Swirl, Line Vortex, and Their Stabilities.

It is important to consider the dynamics and stability of the basic laminar flow, a part of which may become turbulent. In swirling flows the basic laminar flow may enhance, diminish, or even quench turbulence in its interior. These, sometimes strong, stabilizing or destabilizing effects must be considered when postulating turbulent stresses in these flows.

The equation of motion for a swirl or a line vortex is

$$\frac{\partial}{\partial t} r^2 u_\theta = \nu \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} (u_\theta / r), \tag{4}$$

where ν is the kinematic viscosity.

A swirl of finite M diffuses out due to viscosity and shares its angular momentum with the surrounding fluid which is set into motion. Its Reynolds number is $(M/t)^{\frac{1}{2}}/\nu$. A known self-similar swirl is

$$\Gamma = 2\pi r u_\theta = \frac{M}{2\nu t} \left(\frac{r^2}{4\nu t} \right) \exp\left(-r^2/4\nu t\right). \tag{5}$$

In a line vortex $u_\theta \simeq \Gamma_o / 2\pi r$ for $r \to \infty$ and the rate of change of angular momentum,

$$2\pi \int_o^\infty \frac{\partial}{\partial t} r^2 u_\theta dr = -2\nu \Gamma_o . \tag{6}$$

This rate is independent of the detailed distribution of the vorticity or u_θ in the interior of the vortex. If we assume finite ν then there are stresses but no net force on a fluid element in the potential flow surrounding the core where most of the vorticity resides. The angular momentum is lost from the interior of the vortex through the potential flow region to the region $r \to \infty$. The core over which most, say 95%, of the total vorticity or Γ_o is distributed grows and so does its angular momentum. This is due to infinite angular momentum surrounding any finite though growing interior region. The Reynolds number is Γ_o/ν .

A known self-similar solution for the line vortex is

$$\Gamma = 2\pi r u_\theta = \Gamma_o \left[1 - \exp(-r^2/4\nu t)\right] \tag{7}$$

We may combine (5) and (7) to get a vortex-swirl combination

$$\Gamma = \Gamma_o \left[1 + \left\{ \frac{M}{2\nu t\Gamma_o} \left(\frac{r^2}{4\nu t}\right) - 1 \right\} \exp(-r^2/4\nu t)\right] \tag{8}$$

In a line vortex (7) shows that the circulation increases monotonically with radius reaching a constant value Γ_o . In a swirl total vorticity is zero and Γ increases and then decreases to zero for large r. In a line-vortex-swirl example given by (8) Γ overshoots Γ_o , then decreases to Γ_o for $r \to \infty$. This overshoot decreases and becomes relatively insignificant with time and may be considered as a decaying *'initial' disturbance.*

This is a special case of a general result. If the initial vorticity is axisymmetric and is distributed over a finite area around the origin then its subsequent distribution can be found using (4). After a long time vorticity and velocity distributions will appear as if the total vorticity were originally at the origin, any initial vorticity distribution of opposite signs and zero total value having no significant influence. If, however, the total vorticity is zero, then the initial distribution of the vorticity determines the subsequent state of the vorticity and the velocity.

We may look at the situation from the point of view of dynamics. The angular momentum associated with a finite total vorticity is infinite and it is finite for a distribution of vorticity of opposite signs such that the total vorticity or Γ_0 is zero. As time progresses the former will dominate the latter. We are assuming initially concentrated vorticity near the axis.

These well known results are presented to contrast some properties of laminar flows with corresponding properties of the flows with the same overall parameters and in turbulent state. For example, in a laminar line-vortex-swirl the dynamics of any region of overshoot where $\Gamma > \Gamma_0$ become in time unimportant to the dynamics of the main vortex. While in a turbulent line-vortex-swirl Govindraju and Saffman (1971) assert that at high Reynolds numbers the overshoot, $\Gamma > \Gamma_0$, is the main growth mechanism of a turbulent vortex.

The virtual origin of the time t for a line-swirl and line-vortex may not be the same. This becomes insignificant as t becomes large. However, for both small and large t we define a line-vortex-swirl as a flow which behaves like a vortex for sufficiently large r and for some finite range of r, $d\Gamma^2/dr < 0$. This slight generalization is necessary for the purpose at hand, which is to study the stability and turbulence in such flows.

An important criterion based on analysis assuming inviscid flow for stability of swirling flows with <u>one</u> velocity component u_θ (r,t) is, (Rayleigh 1916), (Chandrasekhar 1961),

$$d\Gamma^2/dr > 0. \tag{9}$$

A line vortex is stable at all times. A swirl is unstable. A line-vortex-swirl is unstable and its interior may become turbulent at a sufficiently high Reynolds number. However, as the time progresses the flow will tend to stabilize and production of turbulence will diminish towards zero and the turbulence already created will decay due to viscosity. In practice flows approximating a line-vortex are used to stabilize unstable fluid configurations such as vortex stabilized electric arc, (Chow and Uberoi 1972). Experiments lend support to the above criterion (9) without regard to any limitations of the cited stability theories.

We have conducted a simple experiment to check the stability considerations. Water was injected tangentially all along the inner wall of a transparent vertical cylindrical vessel of 30 cm diameter, 50 cm long, with a central drain in its flat bottom and nearly full of water. After the vortex was set up the drainage was reduced in steps after each repetition of the experiment described below. There was no drainage during the last experiment. The fluid in the center was made turbulent by stirring it randomly or by spinning a 0.6 cm diameter rod spanning the entire length of the cylinder in the direction of the main vortex or opposing it. The turbulent flow was made visible by painting the stirrer or the rod with water soluble ink. Care was taken to limit the disturbance to a short time period, so as to confine the initial disturbance to a cylindrical region of about 3 cm diameter, which may be considered small in size compared with the size of the main vortex. The Reynolds number Γ_o/ν of the vortex was about 10^3. In every case the initial turbulence decayed and was not sustained at the expense of the energy of the relatively slowly changing main motion of the vortex.

In another experiment at $\Gamma_o/\nu = 7.8 \times 10^4$ we have made detailed velocity measurements in a trailing vortex when it approached a line vortex for a short distance, i.e. $u_z \approx u_o$. The existing turbulence in the vortex almost disappeared (Singh and Uberoi 1976). This further substantiates our claim that turbulence cannot be sustained in a line vortex.

3. Turbulent Line-Swirl, Line-Vortex, and Their Combination

A line-swirl is unstable according to the criterion (9) based on the assumption of an inviscid fluid. At sufficiently high Reynolds numbers it becomes unstable, which is consistent with experience. We have recently developed the structure of a turbulent swirl based on plausible assumptions, (Uberoi, 1977a).

A line-vortex is stable unless it has a swirl superimposed on it; i.e., there is a finite radial region where $d\Gamma^2/dr < 0$. Turbulence will develop in the nature of an initial disturbance and decay at a faster rate than the asymptotic rate of growth of the vortex core. However, there are several theories of sustained turbulence in a line-vortex. They assume that there exists three-dimensional turbulence. However, there is only one mean velocity component $u_\theta\ (r,t)$. The equation governing a turbulent line-swirl, line-vortex, or their combination is

$$\frac{\partial}{\partial t}\ (r^2 u_\theta) = \frac{\partial}{\partial r}\ r^2\ (-\overline{u_r' u_\theta'} + \nu r\ \frac{\partial}{\partial r}\ \frac{u_\theta}{r}) \tag{10}$$

where $-\ \rho\overline{u_r' u_\theta'}$ is the turbulent stress.

We note that the equation (6) for the rate of change of angular momentum of a turbulent line-vortex or line-vortex-swirl is the same as for the laminar case. Since any turbulence is confined to the vortex core, it does not affect the ultimate transfer of angular momentum through the potential flow surrounding the vortex core.

Squire (1965) was the first to consider a vortex with a turbulent core. In effect, he assumed that "turbulent" kinematic viscosity,

$$\nu_t = \alpha\Gamma_o \tag{11}$$

where α is a constant. The solution is given by (7) with ν replaced by $\alpha\Gamma_o$. Since the turbulence is confined to a finite radius, ν_t should vanish as $r \to \infty$. Squire's assumption allows far too much angular momentum to escape to infinity which is determined by (6).

We may try to save Squire's solution by stipulating that it is only valid for a vortex in the presence of uniform atmospheric turbulence where a constant ν_t may be used. However, there are serious difficulties with this artifice. A constant turbulent kinematic viscosity due to atmospheric turbulence has nothing to do with Γ_o which is associated with the vortex. Let the atmospheric turbulence be strong enough to interact with the vortex. In the potential part of the vortex, due to spatially varying rate of strain, the interaction would vary spatially and with time. Consequently ν_t cannot be assumed constant. Further, we cannot assume that the entire vortex interacts significantly with the atmospheric turbulence, and the outer flow is still potential with $u_\theta = \Gamma_o/2\pi r$ for all time.

Hoffman and Joubert (1963) considered radial transfer of angular momentum in the turbulent core of a line-vortex. Using certain assumptions they concluded that the circulation varies logarithmically with radial distance in the region of maximum u_θ . They failed to show how the momentum is transferred to large radial distances; this transfer is determined by viscosity and is given by (6).

The total rate of change of angular momentum is due to viscosity and cannot exceed that given by (6). If we insist on the growth of the core, i.e., decrease of its swirl velocities faster than that caused by viscosity, then the outer flow must speed up, since the total angular momentum must be conserved except for a small loss due to viscosity. It follows that the circulation in the region of potential flow where the flow speeds up must exceed or over-shoot Γ_o . Various elaborate theories have been developed to "prove" the existence of a circulation over-shoot. In fact, the over-shoot is a direct consequence of the insistence mentioned above, which may take many different forms.

Govindaraju and Saffman (1971) assume that $r^2 \overline{u_r' u_\theta'}$ and ru_θ are functions of $r/t^{\frac{1}{2}}$ and (4) becomes a total differential equation. The insistence is contained in this functional dependence; i.e., the maximum value of u_θ must decrease and the core size must increase faster than would be caused by viscosity.

Macagno and Macagno (1975) assume that if ε is the mean rate of strain then the turbulent kinematic viscosity

$$\nu_t = \alpha + \frac{\beta|\varepsilon|}{2} \tag{12}$$

where α and β may vary with space and time but are taken to be constant in their analysis. The quantity α here is not related to that in (11). The above is supposed to include vortex generated and atmospheric turbulence. In accordance with the above discussion of Squire's work α must equal ν . The equation (12) allows turbulent stresses and hence production of turbulence in the outer potential flow where ε is finite.

Theories discussed here and other such theories were based on the unjustifiable belief that a line-vortex is equivalent to a trailing vortex where t is replaced by z/u_o. The results of the analyses were compared with experimental observation of trailing vortices.

Measured data on trailing vortices have been fitted to Squire's solution for a line-vortex, although the value of the constant α varies from case to case (Rose and Dee 1963; McCormick et al 1968). Owen (1970) has given an explanation for the variations of α with Γ_o / ν. This does not remove the fundamental objections raised above.

4. Laminar Trailing Swirl, Trailing Vortex, and Their Instabilities

The equation governing u_θ in these flows is

$$\frac{\partial}{\partial z} (u_o + u_z) u_\theta r^2 + \frac{\partial}{\partial r} (u_r u_\theta r^2) = \nu \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} \left(\frac{u_\theta}{r}\right) , \qquad (13)$$

where u_o is the constant prevailing velocity along the swirl axis. In the core of these flows the swirl velocity u_θ decreases with downstream distance z and therefore $\frac{\partial p}{\partial z} > 0$. The flow in the core diverges, causing significant radial and some associated axial convection of angular momentum. We illustrate this by examining the total rate of change of angular momentum in a trailing vortex. Integrating (13), we have

$$2\pi u_o \int_o^\infty \frac{\partial}{\partial z} u_\theta r^2 dr = \underbrace{-2\pi \frac{\partial}{\partial z} (u_z u_\theta) r^2 dr - \left[u_r r\right]_{r \to \infty} . \Gamma_o}_{\text{convection}} \underbrace{- 2\nu\Gamma_o}_{\text{diffusion}} \qquad (14)$$

Batchelor (1964) has calculated axial flow in a trailing vortex, neglecting radial flow. He assumes that all velocities are small compared with u_o, so that (13) becomes

$$u_o \frac{\partial}{\partial z} u_\theta r^2 = \nu \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} \left(\frac{u_\theta}{r}\right) , \tag{15}$$

which replaces a trailing vortex with a line-vortex. The velocity u_θ is given by (7) with t replaced by z/u_o. Using this u_θ pressure is calculated from the approximate equation

$$\rho\, u_\theta^2 / r = \frac{\partial p}{\partial r} . \tag{16}$$

Axial velocity u_z is calculated using this pressure and the equation

$$\rho\, u_o \frac{\partial}{\partial z} u_z = - \frac{\partial p}{\partial z} + \rho\nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z . \tag{17}$$

The result is

$$u_z = - \frac{\Gamma_o^2}{32\pi^2 \nu z} \ln \frac{zu_o}{\nu} \exp(-\xi) + \frac{\Gamma_o^2}{32\pi^2 z} Q_2(\xi) - \frac{Lu_o^2}{8\nu z} \exp(-\xi), \tag{18}$$

where

$$\xi = u_o r^2 / 4\nu z \tag{19}$$

The function Q_2 and the constant L are not of interest here. We have calculated the radial and associated axial convection as

$$\Gamma_o \cdot (u_r r)_{r \to \infty} = -2 \int_o^\infty \frac{\partial}{\partial z} u_z \Gamma r \, dr = \Gamma_o^3 / 16\pi^2 z u_o \tag{20}$$

The importance of the neglected convection of angular momentum can be expressed as the ratio

$$\text{convection/diffusion} = \int_0^\infty \frac{\partial}{\partial z} u_z \Gamma r dr + \Gamma_o \cdot (u_r r)_{r \to \infty} / 2\nu\Gamma_o \tag{21}$$

$$= (\Gamma_o/\nu) \quad (\Gamma_o/64\pi z u_o)$$

$$= (\Gamma_o/\nu) \quad (c/400z)$$

where we have assumed that the trailing vortex is generated by a semi-infinite wing of chord c and $\Gamma_o = cu_o/2$. Since the vortex is essentially a high Reynolds number phenomenon $\Gamma_o/\nu >> 1$ and z/c must be large to make this ratio much smaller than unity. In general radial and associated axial convection of angular momentum cannot be neglected.

Moore and Saffman (1973) have calculated axial velocity in the core of a vortex for which at $z = o$, $u_\theta = \beta r^{-n}$ where β is a constant and $0 < n < 1$. They also neglected radial and associated axial convection of angular momentum. Using their axial velocity, we find that the requirement for these neglects is that

$$\beta^2 / u_o^2 \left(\frac{\nu z}{u_o} \right)^n << 1 \tag{22}$$

They were concerned with axial velocity during vortex sheet roll up near the wing or small z where the flow is essentially three-dimensional and u_r can not be neglected. The condition (22) may be satisfied at large Z, but then vortex is rolled up and corresponds to Batchelor's case.

The present discussion shows that for those distances of interest from the origin where the vortex is significantly changing the dominant mechanism for a decrease in the swirl velocities in the core of a laminar trailing vortex is radial and associated axial convection of angular momentum.

Another important effect of axial flow or difference in u_z between the core and the surroundings is that a flow becomes unstable which was otherwise stable according to criterion (9), (Uberoi et al. 1972).

A trailing vortex is further destabilized when a trailing-swirl is added to it such that there is a finite radial distance r for which $d\Gamma^2/dr < 0$, in the same manner as for a line-vortex.

5. Turbulent Trailing Vortex

On dimensional grounds we may write the functional dependence of the circulation Γ (- $2\pi r u_\theta$) as

$$\Gamma/\Gamma_o = \gamma\ (ru_o/\Gamma_o\ ,\ zu_o/\Gamma_o;\Gamma_o/\nu)\ . \qquad (23)$$

All "free" turbulent flows, i.e., without constraining rigid boundaries, are independent of Reynolds number. There has been some speculation (McCormick et al. 1968) that the structure of a turbulent trailing vortex may depend on its Reynolds number Γ_o/ν .

The equation governing u_θ using slender vortex core approximation is

$$\frac{\partial}{\partial z}\ (u_o + u_z)u_\theta r^2 = \frac{\partial}{\partial r}\ r^2\ (-u_r u_\theta\ -\overline{u_r' u_\theta'} + \nu r\ \frac{\partial}{\partial r}\ \frac{u_\theta}{r}\ )\ . \qquad (24)$$

Integrating this equation we have

$$u_o \int_o^\infty \frac{\partial}{\partial z} \Gamma r dr = -\int_o^\infty \frac{\partial}{\partial z} u_z \Gamma r dr - (u_r r)_{r \to \infty} \Gamma_o - 2\nu\Gamma_o \qquad (25)$$

$$= -\int_o^\infty \frac{\partial}{\partial z} u_z (\Gamma - \Gamma_o)\, r dr \qquad - 2\nu\Gamma_o \qquad (26)$$

convection diffusion

where we have assumed that turbulence in the trailing vortex vanishes as $r \to \infty$ and we have made use of the continuity equation in deriving (26). It is well known that in all flows without constraining boundaries the turbulent fluid is separated from non-turbulent fluid by a sharp irregular boundary. Velocity fluctuations decay very rapidly as we move from turbulent into non-turbulent fluid. Further, measurements, Singh (1974), Uberoi (1974), in a trailing vortex show that both $r(\overline{u_\theta'^2})^{\frac{1}{2}}$ and $r(\overline{u_r'^2})^{\frac{1}{2}}$ vanish as $r \to \infty$. Hence $r^2\, \overline{u_r' u_\theta'} \to o$ as $r \to \infty$. We are concerned here with the spread of turbulence which is initially concentrated near the axis of the trailing vortex. In all cases of turbulent flow thus far experimentally investigated, the irregular front separating the turbulent from non-turbulent fluid propagates at a finite rate rather than diffusing to infinity. The vanishing of $r^2\, \overline{u_r' u_\theta'}$ is consistent with all known experimental facts about the trailing vortex and the turbulent flows in general. Assuming that $r^2\, \overline{u_r' u_\theta'}$ is finite as $r \to \infty$ would lead us to the same difficulty as Squire's work (1965). Angular momentum would escape to infinity due to turbulent stresses far in excess of that allowed by laminar viscosity of the fluid.

It follows from (26) that the radial and associated axial convection of angular momentum are important if the turbulent vortex grows faster than the laminar vortex and there is no overshoot or $\Gamma \not> \Gamma_o$. The experiments determine the magnitude of the relative importance of convection versus diffusion.

The velocities u_θ and u_z in a turbulent trailing vortex behind an airfoil have been measured by Singh (1974) and Uberoi (1974) at $\Gamma_o/\nu = 2.1 \times 10^4$. Unfortunately, the convection of angular momentum cannot be accurately calculated from measured u_z . We have calculated the first and the last terms in (25) using measured u_θ and thus determined that diffusion is about a percent or so of the convection of the angular momentum. Therefore, the dynamics of turbulent trailing vortex are independent of Γ_o/ν at least at the Reynolds number of the experiment and above. In the literature the effect of slowly decaying and different initial conditions in different experiments may have been confused with the effect of Reynolds number on the structure of these vortices. The terms involving ν may be neglected in (23) through (26) (Uberoi 1977b); thus

$$\Gamma/\Gamma_o = \gamma(ru_o/\Gamma_o , zu_o/\Gamma_o) \quad \text{and} \tag{27}$$

$$u_o \frac{\partial}{\partial z} u_\theta^2 r^2 = \frac{\partial}{\partial r} r^2(-u_r u_\theta - \overline{u_r' u_\theta'}) - \frac{\partial}{\partial z} u_z u_\theta r^2 \tag{28}$$

In order to proceed further we may write the equations governing u_r and u_z and assume enough relations among the independent variables so that their number equals the number of equations. Instead we propose an elemental theory which incorporates the mechanism of vortex changes discussed above. We look for a solution such that Γ/Γ_o is a function of a single variable

$$\eta = \left(\frac{r\,u_o}{\Gamma_o}\right)^2 \left(\frac{\Gamma_o}{zu_o}\right)^n \tag{29}$$

We assume that the total radial and associated axial convection of angular momentum

$$\int_0^\infty \left[\frac{\partial}{\partial r} r^2 \left(u_r\, u_\theta + \overline{u_r' u_\theta'}\right) + \frac{\partial}{\partial z} r^2 u_z u_\theta\right] dr \sim \left(\frac{\Gamma_o}{zu_o}\right)^m \cdot \Gamma_o^2 \tag{30}$$

The terms on the right hand side of (28) are only significant in the core where $\Gamma - \Gamma_o$ is significantly different from zero. The sign of these terms should not depend on the sign of $\Gamma - \Gamma_o$ and they should have proper dependence on r as $r \to 0$. On these bases and in light of discussion of the physical phenomena we propose that

$$\frac{\partial}{\partial r} r^2 \left(u_r u_\theta + \overline{u_r' u_\theta'}\right) + \frac{\partial}{\partial z} r^2 u_z u_\theta = n\, a \left(\frac{\Gamma_o}{zu_o}\right)^m (\Gamma - \Gamma_o)^2 \frac{\eta^2}{r} \exp(b\eta) \tag{31}$$

where a and b are constants. The factor $\exp(b\eta)$ is in recognition of the fact that the turbulent core at any z is of finite size and the expression (31) should rapidly approach zero as we go from turbulent core to the outside non-turbulent fluid.

Using (31) the governing equation (28) becomes

$$u_o \frac{\partial}{\partial z} \Gamma r = -n\, a \left(\frac{\Gamma_o}{zu_o}\right)^m (\Gamma - \Gamma_o)^2 \frac{\eta^2}{r} \exp(b\eta) \tag{32}$$

and for $m = 1 - n$ we have

$$\frac{d\gamma}{d\eta} = a(1-\gamma)^2 \exp(b\eta) \tag{33}$$

The solution is

$$\gamma = 1 - 1/(1 + \frac{a}{b} \left[\exp(b\eta) - 1\right]) \tag{34}$$

The comparison of this expression with observations, Singh (1974), Uberoi (1974) is shown in figure 1, where a = 150 and b = 10.

6. Final Stages in Vortex Decay

If in (30) $m > 0$ then relative to diffusion the importance of convection of angular momentum decreases as $z \to \infty$. Independent of the theory proposed here, let us assume that the convection is negligable compared with the diffusion and the core size continues to increase at least at the rate given by the diffusion as $z \to \infty$. With increasing core size and no radial and associated axial convection of angular momentum, the axial velocity becomes unimportant. We claim that under these conditions the flow becomes stable and no sustained turbulence is possible. See section 2 above.

In studies of laminar and turbulent line vortices self-similarity is often assumed (Squire 1965, Govindraju and Saffman 1971). Consider the following form

$$\Gamma/\Gamma_o = \gamma(r^2/ct^n) = \gamma(\xi) \tag{35}$$

Integrating (10) and using (35) we have

$$2\pi\int_o^\infty \frac{\partial}{\partial t} u_\theta r^2 dr = \frac{-n\Gamma_o t^{n-1} c}{2} \int_o^\infty \gamma'\xi d\xi = -2\nu\Gamma_o \tag{36}$$

It follows that $n = 1$ and $c \sim \nu$. Using (35) the expression for the rate of change of kinetic energy of mean motion becomes

$$\pi\int_0^\infty \frac{\partial}{\partial t} u_\theta^2 r \, dr = -\Gamma_o^2/8\pi t \tag{37}$$

It may be shown that $\Gamma_o^2/8\pi t$ is the rate of viscous dissipation of kinetic energy of a laminar line vortex given by (7). In a line vortex with a turbulent core, the rate of decrease of kinetic energy should exceed that in a laminar line vortex with the same over-all parameters. Hence, even if a turbulent line vortex exists it cannot have the form given by (35). We may say that the virtual origins of t are different for laminar and turbulent line vortices. However, this becomes unimportant as $t \to \infty$.

7. Discussion

We have shown that all previous theories of decay of trailing vortices are deficient or just plain wrong since they are based on untenable hypotheses. We find the dominant physical phenomena of radial and associated axial convection of angular momentum and the role of axial velocity in sustaining turbulence in the vortex core.

A theory for a turbulent trailing vortex is presented which satisfies the requirement of turbulent theories. We propose an expression for the dominant physical phenomena which is consistent with the basic equations and the results agree with observations.

Reasonably complete discussion is presented because of confusion in this field and to provide suggestions for further experimental work in turbulent vortices. It is necessary to conduct more extensive experiments to accurately determine the values for m and n. Once we have accurate measurements of u_z then we may use the following relation to independently determine m, from (31) we have

$$\int_o^\infty \frac{\partial}{\partial z} u_z(\Gamma-\Gamma_o) r dr = na\left(\frac{\Gamma_o}{zu_o}\right)^m \int_o^\infty \left(\frac{\Gamma-\Gamma_o}{\Gamma_o}\right)^2 \eta \exp(b\eta) d\eta \qquad (38)$$

In the past there have been no guiding physical processes or theories which could help evaluate various devices and methods for amelioration of vortex wake problem and its influence on the operation of airplanes which may interact with trailing vortices of other airplanes. It is hoped that present theory and discussion of physical phenomena will provide such guidance.

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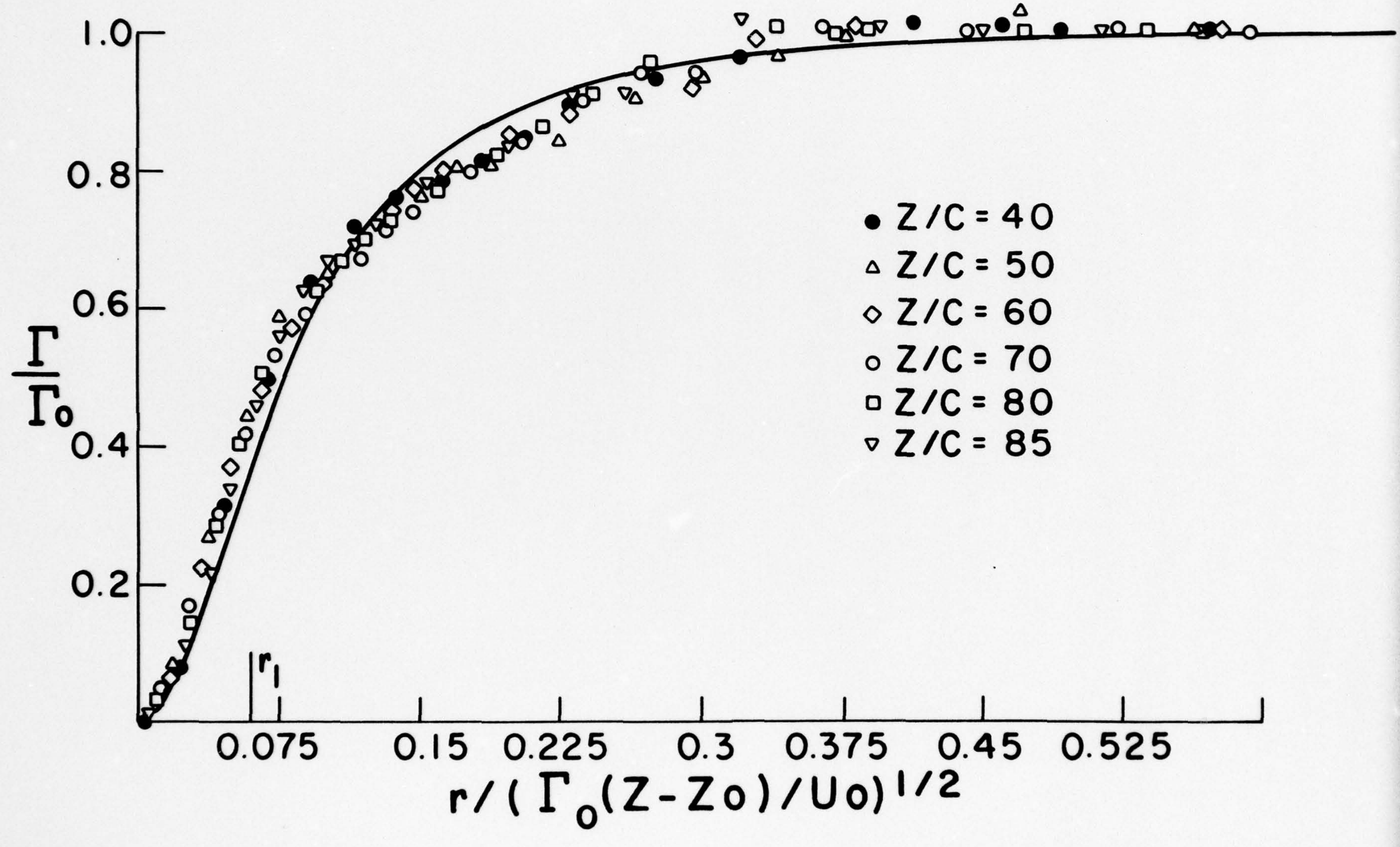

1.0
0.8
0.6
0.4
0.2
Γ/Γo
r1
0.075
0.15
0.225
0.3
0.375
0.45
0.525
r/(Γo(Z-Zo)/Uo)1/2
Z/C = 40
Z/C = 50
Z/C = 60
Z/C = 70
Z/C = 80
Z/C = 85

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APPENDIX

We have used Batchelor's work (1964) to estimate the radial convection and the associated axial convection of angular momentum. His work has been criticized by Tam (1973), which in turn has been criticized by Herron (1974). We can estimate the terms neglected in the linear theory directly from the equations.

The linearized equation of motion for u_z is

$$u_o \frac{\partial}{\partial z} u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2}{\partial r_3} + \frac{1}{r}\frac{\partial}{\partial r}\right)u_z \qquad \text{A1}$$

In the linear theory it is further assumed that the pressure is given by the equation

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} \qquad \text{A2}$$

where $2\pi u_\theta r = \Gamma_o(1-e^{-\eta})$ and $\eta = u_o r^2/4\nu z$. This leads to

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \left(\frac{\Gamma_o}{2\pi}\right)^2 \frac{u_o}{8\nu z^2}\,(P\eta)' \qquad \text{A3}$$

where

$$P(\eta) = \int_\eta^\infty \frac{(1-e^{-t})^2}{t^2}\,d\eta \qquad \text{A4}$$

For the sources given by A3 the diffusion equation A1 has no self-similar or asymptotic solution (see below). Let us assume that at z_o

$$u_z(r, z_o) = 0 \qquad \text{A5}$$

Due to the pressure gradient u_z will, of course, change. This is a reasonable initial condition since the main interest is the change u_z due to the prescribed pressure gradient.

The radial convection and associated axial convection of angular momentum at z_o is given by (14), thus

$$\int_o^\infty \left[\frac{\partial}{\partial z} u_z (\Gamma - \Gamma_o)\right]_{z_o} r\,dr = \int_o^\infty \left[(\Gamma - \Gamma_o)\frac{\partial u_z}{\partial z}\right]_{z_o} rdr$$

$$= \frac{\Gamma_o^3}{16\pi^2 z_o u_o} \int_o^\infty e^{-\eta} (P\eta)' d\eta$$

$$\simeq \frac{\Gamma_o^3}{16\pi^2 z_o u_o} \cdot \left(\frac{1}{2}\right) \qquad \text{A6}$$

where we have made use of A1 - A5 and the value of the definite integral is approximately 1/2. The ratio (see eq. 21)

$$\text{convection/diffusion} \simeq (\Gamma_o/\nu)(\Gamma_o/64\pi z_o u_o) \qquad \text{A7}$$

which agrees approximately with (21).

In order to examine the general solution of A1 we consider the change in the integrated velocity which is, of course, determined by the sources given by A3. Integrating A1 and using A3 and A4 for large η, we have

$$-\frac{\partial}{\partial z}\int_0^\infty u_z r dr = \left(\frac{\Gamma_o}{2\pi}\right)^2 \frac{1}{4u_o z} \left[P\eta \right]_o^\infty$$

$$= \frac{\Gamma_o^2}{16\pi^2 u_o z} \qquad \text{A8}$$

or

$$-\int_o^\infty u_z r \, dr = \frac{\Gamma_o^2}{16\pi^2 u_o z} \ln z/z_o \qquad \text{A9}$$

where we have assumed that there is no velocity deficit at z_o. The integrated velocity deficit given by A9 continues to increase with z and can be made to have any value by suitable choice of z_o. A particular self-similar solution, namely

$$u_z = \frac{\Gamma_o^2}{8\nu z} e^{-\eta} \int_o^\infty P(t)e^t dt \qquad \text{A10}$$

was proposed by Tam (1973) and gives infinite integrated axial velocity deficit as it is should in view of our discussion.

For the sources given by A3 the only properly posed problem is to prescribe an initial $u_z(r,z_o)$ and A1 will determine the subsequent development of u_z. There is no solution which is either self-similar, or independent of $u_z(r,z_o)$ and z_o (depending only on the integrated $u_z(r,z_o)$).

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Dynamics of an infinitely long one-dimensional vortex and a swirl are compared with dynamics of semi-infinitely long trailing vortex and trailing swirl. With increasing distance, the change in axial velocity difference between the core of the trailing-vortex and the surrounding region causes radial convection and some associated axial convection of angular momentum. In laminar or turbulent trailing vortices, this is the dominant mechanism for decrease in velocities of swirl in the core and corresponding growth of the core. Axial velocity difference between the core of the trailing-vortex and

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SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*) the surrounding region is necessary for the sustenance of turbulence in the vortex core. A theory of turbulent trailing-vortex is developed on the basis of these mechanisms and the results are compared with experimental observations.